Problem 1) $n_a(\omega) = \sqrt{\mu_a(\omega)\varepsilon_a(\omega)} = \sqrt{\varepsilon_a(\omega)}$. Similarly, $n_b(\omega) = \sqrt{\mu_b(\omega)\varepsilon_b(\omega)} = \sqrt{\varepsilon_b(\omega)}$.

a) $\boldsymbol{k}^{(\mathrm{i})} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z^{(\mathrm{i})}\hat{\boldsymbol{z}} = n_a(\omega)(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})$.

$\boldsymbol{k}^{(\mathrm{r})} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z^{(\mathrm{r})}\hat{\boldsymbol{z}} = n_a(\omega)(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})$.

$\boldsymbol{k}^{(\mathrm{t})} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z^{(\mathrm{t})}\hat{\boldsymbol{z}} = n_a(\omega)(\omega/c)\sin\theta\,\hat{\boldsymbol{x}} + k_z^{(\mathrm{t})}\hat{\boldsymbol{z}}$.

The dispersion relation $k_x^2 + k_z^2 = (\omega/c)^2 n_a^2(\omega)$ is used here. Also invoked is the generalized Snell's law.

b) Dispersion relation: $\boldsymbol{k}\cdot\boldsymbol{k} = k^2 = (\omega/c)^2\mu_b(\omega)\varepsilon_b(\omega) \;\rightarrow\; k_x^2 + k_z^2 = (\omega/c)^2 n_b^2(\omega)$

$$\rightarrow\quad k_z^{(\mathrm{t})} = \pm\sqrt{(\omega/c)^2 n_b^2(\omega) - k_x^2} \quad\rightarrow\quad k_z^{(\mathrm{t})} = -\mathrm{i}(\omega/c)\sqrt{n_a^2(\omega)\sin^2\theta - n_b^2(\omega)}.$$

Since $\theta > \theta_c$, we have $n_a\sin\theta > n_a\sin\theta_c = n_b$. Therefore, k_z^2 is negative, which makes its square root imaginary. We have chosen the negative sign for $k_z^{(\mathrm{t})}$ to ensure the exponential decay (as opposed to growth) of the evanescent field away from the interface (i.e., as $z \to -\infty$). This is now guaranteed, since the z-dependent factor in the expression of the fields, namely,

$$\exp(\mathrm{i}k_z^{(\mathrm{t})}z) = \exp[(\omega/c)\sqrt{n_a^2(\omega)\sin^2\theta - n_b^2(\omega)}\,z],$$

approaches zero when $z \to -\infty$.

c) $\boldsymbol{B}^{(\mathrm{t})}(\boldsymbol{r},t) = \mu_0\mu(\omega)H_{0y}\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{r} - \omega t)]$.

For $\boldsymbol{\nabla}\cdot\boldsymbol{B}^{(\mathrm{t})} = \mathrm{i}\boldsymbol{k}^{(\mathrm{t})}\cdot\mu_0\mu(\omega)H_{0y}\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{r} - \omega t)]$ to vanish it is necessary to have $\boldsymbol{k}^{(\mathrm{t})}\cdot\hat{\boldsymbol{y}} = (k_x\hat{\boldsymbol{x}} + k_z^{(\mathrm{t})}\hat{\boldsymbol{z}})\cdot\hat{\boldsymbol{y}} = 0$, which obviously holds, since $\boldsymbol{k}^{(\mathrm{t})}$ has no y-component.

d) $\boldsymbol{\nabla}\times\boldsymbol{H} = \partial_t\boldsymbol{D} \quad\rightarrow\quad \mathrm{i}\boldsymbol{k}^{(\mathrm{t})}\times\boldsymbol{H}_0^{(\mathrm{t})} = -\mathrm{i}\omega\varepsilon_0\varepsilon_b(\omega)\boldsymbol{E}_0^{(\mathrm{t})}$

replacing ε_0 with $1/(cZ_0)$

$$\rightarrow\quad (k_x\hat{\boldsymbol{x}} + k_z^{(\mathrm{t})}\hat{\boldsymbol{z}})\times H_{0y}\hat{\boldsymbol{y}} = -(\omega/cZ_0)n_b^2(\omega)(E_{0x}\hat{\boldsymbol{x}} + E_{0y}\hat{\boldsymbol{y}} + E_{0z}\hat{\boldsymbol{z}}).$$

Equating the x, y, and z components appearing on the two sides of the above equation, we find

$$\mathrm{i}(\omega/c)\sqrt{n_a^2(\omega)\sin^2\theta - n_b^2(\omega)}H_{0y} = -(\omega/cZ_0)n_b^2(\omega)E_{0x},$$

$$E_{0y} = 0,$$

$$(\omega/c)n_a(\omega)\sin\theta\,H_{0y} = -(\omega/cZ_0)n_b^2(\omega)E_{0z}^{(\mathrm{t})}.$$

Further simplification now yields

$$E_{0x} = -\mathrm{i}Z_0H_{0y}\sqrt{n_a^2(\omega)\sin^2\theta - n_b^2(\omega)}/n_b^2(\omega),$$

$$E_{0z} = -Z_0H_{0y}n_a(\omega)\sin\theta/n_b^2(\omega).$$

Complete expressions for the evanescent $\boldsymbol{E}$ and $\boldsymbol{H}$ fields may finally be written down, as follows:

$$\begin{aligned}\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) &= (E_{0x}\hat{\boldsymbol{x}} + E_{0z}\hat{\boldsymbol{z}})\exp[\mathrm{i}(k_xx + k_z^{(t)}z - \omega t)]\\ &= -(Z_0H_{0y}/n_b)\left[\mathrm{i}\sqrt{(n_a\sin\theta/n_b)^2 - 1}\,\hat{\boldsymbol{x}} + (n_a\sin\theta/n_b)\hat{\boldsymbol{z}}\right]\\ &\quad\times\exp\left[(n_b\omega/c)\sqrt{(n_a\sin\theta/n_b)^2 - 1}\,z\right]\exp[\mathrm{i}(k_xx - \omega t)].\end{aligned}$$

$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = H_{0y}\hat{\boldsymbol{y}}\exp\left[(n_b\omega/c)\sqrt{(n_a\sin\theta/n_b)^2-1}\,z\right]\exp[\mathrm{i}(k_x x-\omega t)].$$

e) In the absence of free charges (i.e., $\rho_{\text{free}}=0$), Maxwell's 1st equation (within the transmission medium) reduces to $\boldsymbol{\nabla}\cdot\boldsymbol{D}^{(\mathrm{t})}=\varepsilon_0\varepsilon_b(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{E}^{(\mathrm{t})}=0$. For the evanescent wave, the satisfaction this equation requires that $\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{E}_0^{(\mathrm{t})}$ vanish. This constraint is readily satisfied, since we have

$$\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{E}_0^{(\mathrm{t})} = k_x E_{0x} + k_z^{(\mathrm{t})}E_{0z} = (n_a\omega/c)\sin\theta\left[-\mathrm{i}(Z_0H_{0y}/n_b)\sqrt{(n_a\sin\theta/n_b)^2-1}\right]$$
$$+\left[-\mathrm{i}(\omega/c)\sqrt{n_a^2\sin^2\theta-n_b^2}\right]\left(-Z_0H_{0y}n_a\sin\theta/n_b^2\right)=0.$$

As for Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial_t\boldsymbol{B}$, we must show that $\boldsymbol{k}^{(\mathrm{t})}\times\boldsymbol{E}_0^{(\mathrm{t})}=\omega\mu_0\mu(\omega)\boldsymbol{H}_0^{(\mathrm{t})}$.

$$k_z^{(\mathrm{t})}E_{0x}-k_xE_{0z} = \left[-\mathrm{i}(\omega/c)\sqrt{n_a^2\sin^2\theta-n_b^2}\right]\left[-\mathrm{i}Z_0H_{0y}\sqrt{n_a^2\sin^2\theta-n_b^2}/n_b^2\right]$$
$$-(n_a\omega/c)\sin\theta\,(-Z_0H_{0y}n_a\sin\theta/n_b^2)$$
$$=-(\omega/c)[(n_a\sin\theta/n_b)^2-1]Z_0H_{0y}+(\omega/c)(n_a\sin\theta/n_b)^2Z_0H_{0y}$$
$$=(\omega/c)Z_0H_{0y}=\omega\mu_0H_{0y}.$$

f) $$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = ½\mathrm{Re}\left[(E_{0x}\hat{\boldsymbol{x}}+E_{0z}\hat{\boldsymbol{z}})\times H_{0y}^*\hat{\boldsymbol{y}}\right]\exp(2|k_z^{(\mathrm{t})}|z)$$
$$=½\mathrm{Re}\left(E_{0x}H_{0y}^*\hat{\boldsymbol{z}}-E_{0z}H_{0y}^*\hat{\boldsymbol{x}}\right)\exp\left[2(\omega/c)\sqrt{n_a^2\sin^2\theta-n_b^2}\,z\right]$$
$$=Z_0|H_{0y}|^2(n_a\sin\theta/2n_b^2)\exp\left[(2n_b\omega/c)\sqrt{(n_a\sin\theta/n_b)^2-1}\,z\right]\hat{\boldsymbol{x}}.$$

Note that the z-component of the time-averaged Poynting vector has disappeared from the above equation since $E_{0x}H_{0y}^*$ is purely imaginary. Also, the Poynting vector has no y-component. The energy flow rate does have a component along the x-axis, which rapidly decays as $z\to-\infty$.

Problem 2) a) From the dispersion relation, the magnitude of the k-vector in free space is found to be $k = \omega/c$. Considering that both $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are in the xz-plane (i.e., $k_y = 0$), we will have

$$\boldsymbol{k}_1 = (\omega/c)(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}}). \tag{1}$$

$$\boldsymbol{E}_1(\boldsymbol{r},t) = \boldsymbol{E}_{01}\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t)] = E_0\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(x\cos\theta + z\sin\theta - ct)]. \tag{2}$$

$$\boldsymbol{H}_1(\boldsymbol{r},t) = \boldsymbol{H}_{01}\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t)]$$
$$= (E_0/Z_0)(-\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\cos\theta + z\sin\theta - ct)]. \tag{3}$$

Similarly,

$$\boldsymbol{k}_2 = (\omega/c)(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}}). \tag{4}$$

$$\boldsymbol{E}_2(\boldsymbol{r},t) = \boldsymbol{E}_{02}\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t)] = E_0\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(x\cos\theta - z\sin\theta - ct)]. \tag{5}$$

$$\boldsymbol{H}_2(\boldsymbol{r},t) = \boldsymbol{H}_{02}\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t)]$$
$$= (E_0/Z_0)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\cos\theta - z\sin\theta - ct)]. \tag{6}$$

b) Considering that $H_y = 0$ and that H_x and H_z do not depend on the y-coordinate, the expression of the curl of $\boldsymbol{H}$ (evaluated in the plane of the sheet at $z = 0$) is simplified, as follows:

$$\boldsymbol{\nabla}\times\boldsymbol{H} = (\partial_z H_x - \partial_x H_z)\hat{\boldsymbol{y}} \cong \left(-\frac{2E_0\sin\theta}{Z_0 d} - \frac{\mathrm{i}E_0\omega\cos^2\theta}{Z_0 c}\right)\hat{\boldsymbol{y}}\,e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}. \tag{7}$$

ordinary differentiation of H_z with respect to x, since H_z is continuous at $z = 0$

$\partial_z H_x \cong \Delta H_x/\Delta z = [H_x(x, z = d/2, t) - H_x(x, z = -d/2, t)]/d$

Since $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 E_0\hat{\boldsymbol{y}}e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)} + P_0\hat{\boldsymbol{y}}e^{\mathrm{i}(\kappa_0 x - \omega t - \varphi_0)}$, equating $\boldsymbol{\nabla}\times\boldsymbol{H}$ of Eq.(7) with $\partial_t\boldsymbol{D} = -\mathrm{i}\omega\boldsymbol{D}(\boldsymbol{r},t)$ reveals that $\kappa_0 = (\omega/c)\cos\theta$.

c) Continuity of $\boldsymbol{E}_\parallel$ is satisfied, as the E-field on both sides of the sheet is $E_0\hat{\boldsymbol{y}}\,e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}$. This is also the E-field inside the sheet, acting on the electric dipoles of the material.

Similarly, the continuity of $\boldsymbol{B}_\perp$ is automatically satisfied, as the perpendicular B-field on both sides of the sheet is seen from Eqs.(3) and (6) to be $\mu_0 H_z = (E_0/c)\cos\theta\,e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}$.

The tangential H-field (i.e., H_x) is discontinuous at the surface of the sheet, being equal to $\pm(E_0/Z_0)\sin\theta\,e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}$ on the left- and right-hand sides, respectively; see Eqs.(3) and (6). Inside the dielectric material, the D-field is $\boldsymbol{D}(\boldsymbol{r},t) = (\varepsilon_0 E_0 + P_0 e^{-\mathrm{i}\varphi_0})\hat{\boldsymbol{y}}e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}$. Considering that, in the absence of free currents (i.e., $\boldsymbol{J}_{\text{free}} = 0$), $\boldsymbol{\nabla}\times\boldsymbol{H} = \partial_t\boldsymbol{D} = -\mathrm{i}\omega\boldsymbol{D}$, and that the sheet thickness d is sufficiently small, we arrive at

$$\frac{2E_0\sin\theta}{Z_0 d} \cong \mathrm{i}\omega(\varepsilon_0 E_0 + P_0 e^{-\mathrm{i}\varphi_0}). \tag{8}$$

The approximate equality in the above equation becomes exact in the limit when $d \to 0$. The near equality in Eq.(8) could also be obtained with the aid of Eq.(7), where the first term on the right-hand side of Eq.(7) dominates the second term when $d \ll c/\omega = \lambda_0/2\pi$.

d) For the incident beam at the location of the sheet (i.e., at $z = 0$), we have

$$\boldsymbol{\nabla}\times\boldsymbol{H}^{(\text{inc})} = \partial_t\boldsymbol{D}^{(\text{inc})} = -\mathrm{i}\omega\varepsilon_0\boldsymbol{E}^{(\text{inc})} = -\left(\frac{\mathrm{i}E_0^{(\text{inc})}\omega}{Z_0 c}\right)\hat{\boldsymbol{y}}\,e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}. \tag{9}$$

The above contribution to the curl of the H-field at $z = 0$ should now be added to Eq.(7). However, for $d \ll c/\omega = \lambda_0/2\pi$, we may ignore this contribution of the incident beam, just as we ignored the second term on the right-hand side of Eq.(7). Consequently, for a sufficiently thin sheet, $\boldsymbol{\nabla} \times \boldsymbol{H}$ will be dominated by the discontinuity in $\boldsymbol{H}_\parallel$ across the sheet produced by the two radiated plane-waves. Given that $\boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon_0 \varepsilon(\omega)(E_0^{(\text{inc})} + E_0)\hat{\boldsymbol{y}} e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}$, application of Maxwell's 2nd equation, $\boldsymbol{\nabla} \times \boldsymbol{H} = \partial_t \boldsymbol{D} = -\mathrm{i}\omega \boldsymbol{D}$, now yields

$$(2E_0 \sin\theta / Z_0 d) e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)} \cong \mathrm{i}\omega\varepsilon_0 \varepsilon(\omega)(E_0^{(\text{inc})} + E_0) e^{\mathrm{i}(\omega/c)(x\cos\theta - ct)}. \tag{10}$$

Solving the above equation for the reflected field amplitude E_0, we find

$$E_0 \cong -\frac{E_0^{(\text{inc})}}{1 + \mathrm{i}[2c\sin\theta/\omega\varepsilon(\omega)d]} \quad \rightarrow \quad E_0/E_0^{(\text{inc})} \cong -\frac{\pi\varepsilon(\omega)d}{\pi\varepsilon(\omega)d + \mathrm{i}\lambda_0 \sin\theta}. \quad \leftarrow \boxed{\lambda_0 = 2\pi c/\omega} \tag{11}$$

e) The transmitted E-field is readily found from Eq.(10), as follows:

$$E_0^{(\text{trans})} = E_0^{(\text{inc})} + E_0 \cong \frac{(2E_0/Z_0 d)\sin\theta}{\mathrm{i}\omega\varepsilon_0\varepsilon(\omega)} \quad \rightarrow \quad E_0^{(\text{trans})}/E_0^{(\text{inc})} \cong \frac{\mathrm{i}\lambda_0 \sin\theta}{\pi\varepsilon(\omega)d + \mathrm{i}\lambda_0\sin\theta}. \tag{12}$$

Digression. Setting $\theta = 45°$, and $\varepsilon(\omega) = \lambda_0/(\sqrt{2}\pi d)$, the reflection coefficient obtained from Eq.(11) will be

$$E_0/E_0^{(\text{inc})} = -1/(1 + \mathrm{i}) = e^{\mathrm{i}3\pi/4}/\sqrt{2}. \tag{13}$$

Similarly, Eq.(12) yields the transmission coefficient, as follows:

$$E_0^{(\text{trans})}/E_0^{(\text{inc})} = \mathrm{i}/(1 + \mathrm{i}) = e^{\mathrm{i}\pi/4}/\sqrt{2}. \tag{14}$$

Both the reflected and transmitted E-field amplitudes are seen to be $1/\sqrt{2}$ times that of the incident E-field. While the reflected E-field is phase-shifted (relative to the incident E-field) by 135°, the relative phase-shift of the transmitted E-field is 45°. The thin dielectric sheet thus exhibits the essential characteristics of a $50/50$ beam-splitter. Note that, for this to hold to a good approximation, the required value of $\varepsilon(\omega)$, namely, $\lambda_0/(\sqrt{2}\pi d)$, may have to be impractically large, given that d needs to be substantially smaller than the incident wavelength. In fact, recalling that $n(\omega) = \sqrt{\varepsilon(\omega)}$, the relation between d and the wavelength λ_0/n inside the dielectric medium will be $\lambda_0/nd = \sqrt{2}\pi n$. For d to be only one-tenth of λ_0/n, it will be necessary to have $n = 2.25$.
